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C^1 HYPERSURFACES OF THE HEISENBERG GROUP ARE N-RECTIFIABLE

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ABSTRACT. We show that C^1 hypersurfaces in the Heisenberg group are countably N-rectifiable. As a corollary, this shows that all C^1_H graphs over the xy-plane are countable N-rectifiable, showing the equivalence of this notion of rectifiability with that of Franchi, Serra Cassano and Serapioni [4] for such surfaces.

1. INTRODUCTION

In this note, we consider two notions of rectifiability for surfaces in the Heisenberg group. First, we review and fix notation. We denote by $\mathbb{H}$ the three dimensional Heisenberg group and let $\mathfrak{h}$ be its Lie algebra. Recall that

$$\mathfrak{h} = span\{X, Y, Z\}$$

where the only nontrivial bracket relation is $[X, Y] = Z$. In this paper, we will use an identification with $\mathbb{R}^3$ for computational purposes. Let $\{x, y, z\}$ be the standard coordinates on $\mathbb{R}^3$. Then,

$$X = \partial_x - \frac{y}{2}\partial_z$$
$$Y = \partial_y + \frac{x}{2}\partial_z$$
$$Z = \partial_z$$

As the exponential map is a diffeomorphism, we will identify $\mathbb{H}$ with $\mathbb{R}^3$ as folows: the triple (a, b, c) denotes the point $e^{a\ X+b\ Y+c\ Z}$. We fix a background Riemannian metric, g, on $\mathbb{H}$, which makes $\{X, Y, Z\}$ an orthonormal basis at each point.

To describe the Carnot-Carathéodory metric on $\mathbb{H}$, we review some definitions.

Definition 1.1. *The horizontal bundle of* $\mathbb{H}$ *is* $H\mathbb{H} = span\{X, Y\}$.

Definition 1.2. *Let* $\mathscr{A}$ *be the space of absolutely continuous paths in* $\mathbb{H}$ *so that, where the derivative exists, it lies in* $H\mathbb{H}$.

Next, we define the standard **Carnot-Carathéodory metric** on $\mathbb{H}$,

$$d_{cc}(m, n) = \inf_{\gamma \in \mathscr{A}} \left\{ \int < \gamma', \gamma' >^{\frac{1}{2}} |\gamma(0) = m, \gamma(1) = n \right\}$$

For ease of computation, we will use the **Carnot gauge**. Recall that if C is a compact set and $m, n \in C$, there exists constants B (depending on C) so that

$$B^{-1} d_{cc}(m, n) \le d(m, n) \le B d_{cc}(m, n) \tag{1}$$

where $d(m, n) = ||m^{-1}n||$ and

$$||(a, b, c)|| = ((a^2 + b^2)^2 + c^2)^{\frac{1}{4}}$$

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See [2] for more details on the relation between the Carnot gauge and the Carnot-Carathéodory metric. We note that left translation in the group and rotations about the z-axis are isometries in both the Carnot-Carathéodory metric and the Carnot gauge.

In [11], the second author introduced the notion of N-rectifiability:

Definition 1. *Let N' be a Carnot group and N be a subgroup of N' with Hausdorff dimension k. A subset E of another Carnot Group (M, d_M) is said to be N-**rectifiable** if there exists U, a positive measure subset of N, and a Lipschitz map $f : U \to M$ such that $\mathscr{H}^k_M(E \setminus f(U)) = 0$. We say E is **countably** N-**rectifiable** if there exist a countable number of pairs of subsets U_i and Lipschitz maps $f_i : U_i \to M$ with $\mathscr{H}^k_M(E \setminus \cup_i f_i(U_i)) = 0$.*

In this definition $\mathscr{H}^k_M$ is the k-Hausdorff measure on M computed with respect to the Carnot-Carathédory metric d_M. In the same paper, the second author showed several local measure theoretic approximability properties of N-rectifiable sets and demonstrated that level sets of C^1_H functions in a Carnot group share the same approximability properties. Moreover, he gave a condition on tangent cones that ensured N-rectifiability of subsets of Carnot groups.

In [4], Franchi, Serra Cassano and Serapioni introduced a different notion of rectifiability in the Heisenberg group (later, the same authors extended this notion to two step groups [3, 5]), $\mathbb{H}$, based on a notion of horizontal regularity:

Definition 1.3. *A function $f : \mathbb{H} \to \mathbb{R}$ is of class C^1_H if Xf and Yf exist and are continuous. A surface, S, given as the level set of a function f is called a C^1_H surface if $f \in C^1_H$.*

We note that while the classes of C^1 and C^1_H surfaces are distinct, if we restrict our attention to graphs over the xy-plane (i.e. surfaces given by $t = f(x, y)$), then the classes are the same. Definition 1.3 gives rise to an alternate definition of rectifiability:

Definition 1.4. *A Cacciopoli set E is $\mathbb{H}$-rectifiable if*

$$\partial E = N \cup \bigcup_{i=1}^{\infty} K_i$$

where N has zero $\mathscr{H}^3_{\mathbb{H}}$ measure and each K_i is a compact subset of a noncharacteristic C^1_H hypersurface.

In this note, we present a more concrete class of N-rectifiable sets which are not covered in [11]. We investigate the N-rectifiability of C^1 hypersurfaces in the three dimensional Heisenberg group.

By explicitly constructing Lipschitz mappings, we show the following theorem:

Theorem 1. *Let S be a C^1 hypersurface in $\mathbb{H}$ and let N be the subgroup given by $\{0, y, z\} \subset \mathbb{H}$. Then S is countably N-rectifiable.*

The subgroup N in the theorem is isomorphic to the metric tangent cone at points on S (see [2] or [10] for a description of the tangent cones to Carnot-Carathéodory metric spaces). Thus, this theorem recovers some of the flavor of its Euclidean counterpart: patches of C^1 surfaces look locally like their tangent approximations. As pointed out in [11], the Euclidean trick of using the projection is insufficient for the Carnot setting as the inverse of a projection is not a Lipschitz map.

As an immediate corollary of to the main theorem in [4] and the observation after defintion 1.4 above, we have:

Corollary 1.5. *If S is a C^1_H graph over the xy-plane in $\mathbb{H}$ then S is countably N-rectifiable. Moreover, any N-rectifiable graph over the xy-plane is $\mathbb{H}$-rectifiable.*

We briefly remark that this investigation is somewhat different than that of Serra Cassano and Kirchheim ([6]) where those authors construct Euclidean $\frac{1}{2}$-Hölder parameterizations of C^1_H surfaces and show that the exponent cannot, in general, be improved. In our treatment, we insist that the domain be equipped with a degenerate metric which is inherited from the Carnot-Carathéodory metric on $\mathbb{H}$.

To describe the mapping geometrically, we make some initial observations. First, the subgroup $N = \{(0, y, z)\}$ is endowed with a restriction of the Carnot metric from $\mathbb{H}$. If we consider N abstractly (as opposed to as a subset of $\mathbb{H}$), we will denote the point $(0, y, z)$ by (y, z). To facilitate checking the Lipschitz condition, we will use the restriction of the Carnot gauge to compute distances. Precisely, if $(y_1, z_1), (y_2, z_2) \in N$,

$$d_N((y_1, z_1), (y_2, z_2)) = \left((y_1 - y_2)^4 + (z_1 - z_2)^2\right)^{\frac{1}{4}}$$

To understand better the construction of the Lipschitz mapping, we note that curves of the form (y, z_0), for fixed z_0, are 1-Lipschitz images of $\mathbb{R}$ while curves of the form (y_0, z), for fixed y_0, are $\frac{1}{2}$-Hölder images of $\mathbb{R}$. Thus, a necessary condition on the mapping is that these curves map to curves of the same class.

Second, if we consider the intersection of the horizontal bundle with the Riemannian tangent plane at a point on S, we have that, generically, there is a single horizontal line contained in each tangent plane. To see this, we compute the Riemannian normal to the surface:

$$N = (Xf)\, X + (Yf)\, Y + (Zf)\, Z$$

We note that the only horizontal vector field (up to a multiple) is given by

$$V = -\frac{Yf}{Xf} X + Y$$

We note that

$$Vf = -\frac{Yf}{Xf} Xf + Yf = 0$$

and so the vector field V is tangential to the surface S and is observably horizontal. The only points where the vector field V would not be well defined (up to a multiple) are the characteristic points, i.e. places where $Xf = Yf = 0$. At points of the charactersitic locus, we have that the entire tangent space is horizontal. Thus, away from the characteristic locus, there exists a horizontal field on the surface S and we may consider integral curves of this vector field. These curves, as they are horizontal, are rectifiable curves (in the usual sense) — locally, they are Lipschitz images of $\mathbb{R}$. Other curves on S are $\frac{1}{2}$-Hölder images of $\mathbb{R}$. From the point of view of understanding the N-rectifiability of the surface, we recall that in [1], Z. Balogh showed that the characteristic locus of a C^1 hypersurface has $\mathscr{H}^3_{cc}$ measure zero (see also the sharp results of Magnani [7–9]). As we may ignore sets of measure zero when discussing N-rectifiability, we may ignore the entire characteristic locus. For the balance of the paper, we consider a noncharacteristic neighborhood of S and fix a particular choice of integral curves of V.

Putting together these two observations shows how to geometrically construct the mapping. On a neighborhood with no characteristic points, consider a smooth curve $\varphi \subset S$, which is transverse to the horizontal curves everywhere (i.e. φ' is not a horizontal vector). Let $\theta_{\varphi(s)}(r)$ be the integral curve of the horizontal vector field passing through $\varphi(s)$. Our

candidate mapping is

$$\begin{aligned}\Psi : N &\to S \subset \mathbb{H}\\(y,z) &\to \theta_{\varphi(z)}(y)\end{aligned}$$

In practice, we will identify a specific curve φ, the intersection of the surface with the plane $y = 0$. The bulk of the paper is showing that this map is locally Lipschitz.

2. Proof of theorem 1

To fix notation, we let $f : \mathbb{H} \to \mathbb{R}$ be a C^1 function and S be the surface given by the level set $f = 0$. Using a suitable left translation, we may assume that $f(\mathbf{0}) = 0$. Moreover, we assume that the origin is not a characteristic point, i.e. $(Xf, Yf)(\mathbf{0}) \neq \mathbf{0}$. Again, by composing with suitable isometries of the Carnot-Carathéodory metric, we may assume that $Xf(\mathbf{0}) > 0$ and $Yf(\mathbf{0}) = 0$. By the continuity of Xf, there exists an open neighborhood U_1 of $\mathbf{0}$ such that for all $q \in U_1$ we have that $Xf(q) > 0$.

Since U_1 is open, there exists an $a > 0$ such that the region

$$\left\{(x,y,z) \in \mathbb{H} \,\middle|\, -a \leq x \leq a,\ -a \leq y \leq a,\ -a - \frac{xy}{2} \leq z \leq a - \frac{xy}{2}\right\}$$

is contained in U_1. Call this region C_1. As C_1 is compact, we have that

- there exists a constant $K > 0$ such that for all $q \in C_1$, $Xf(q) \geq K$.
- there exists a constant $L > 0$ such that for all $q \in C_1$, $|\partial_x f(q)| \leq L$, $|\partial_y f(q)| \leq L$, and $|\partial_z f(q)| \leq L$.
- there exists a constant $M \geq 0$ such that for all $q \in C_1$, $\left|\frac{Yf(q)}{Xf(q)}\right| \leq M$.

Since on the plane $y = 0$ we have that $X = \partial_x$, if follows that $K \leq L$.

Claim 1: There exists a continuous curve $\varphi(z)$ parameterizing the intersection of $f(x,0,z) = 0$ and C_1 with $z \in \left[\frac{-K}{L}a, \frac{K}{L}a\right]$.

We first note that if $L|z| \leq K|x|$ and $x \geq 0$, then $f(x,0,z) \geq 0$:

$$\begin{aligned}f(x,0,z) &= f(x,0,0) + \int_0^z \partial_z f(x,0,t)\,\mathrm{d}t \geq f(x,0,0) - L|z| \geq f(x,0,0) - K|x|\\&= f(0,0,0) + \int_0^x \partial_x f(t,0,0)\,\mathrm{d}t - Kx = f(0,0,0) + \int_0^x Xf\,\mathrm{d}t - Kx\\&\geq f(0,0,0) + Kx - Kx = f(0,0,0) = 0;\end{aligned}$$

and if $L|z| \leq K|x|$ and $x \leq 0$, then $f(x,0,z) \leq 0$:

$$\begin{aligned}f(x,0,z) &= f(x,0,0) + \int_0^z \partial_z f(x,0,t)\,\mathrm{d}t \leq f(x,0,0) + L|z| \leq f(x,0,0) + K|x|\\&= f(0,0,0) + \int_0^x \partial_x f(t,0,0)\,\mathrm{d}t - Kx = f(0,0,0) + \int_0^x Xf\,\mathrm{d}t - Kx\\&\leq f(0,0,0) + Kx - Kx = f(0,0,0) = 0.\end{aligned}$$

Since $\partial_x f = Xf \geq K$ on the intersection of C_1 with the plane $y = 0$, we have that for all $z \in [-a,a]$ there exists at most one $x \in [-a,a]$ such that $f(x,0,z) = 0$. Fix a value of $z \in \left[-\frac{K}{L}a, \frac{K}{L}a\right]$. Note that then $-a \leq -\frac{L}{K}|z|$ and $\frac{L}{K}|z| \leq a$. The map $x \mapsto f(x,0,z)$ on the domain $[-a,a]$ is continuous, and we know from the inequalities above that $f\left(-\frac{L}{K}|z|, 0, z\right) \leq 0$ and $f\left(\frac{L}{K}|z|, 0, z\right) \geq 0$. Thus by the Intermediate Value Theorem there exists $x \in \left[-\frac{L}{K}|z|, \frac{L}{K}|z|\right]$ such that $f(x,0,z) = 0$. Denote this point

$(x, 0, z)$ by $\varphi(z)$. The continuous curve $\varphi(z)$ then parameterizes the intersection of the level set $f(x, y, z) = 0$ with C_1 when $y = 0$ and $z \in \left[-\frac{K}{L}a, \frac{K}{L}a\right]$, proving the claim.

Next, we recall the vector field defined in the introduction

$$V = -\frac{Yf}{Xf}X + Y$$

on the region C_1. Note that V is well-defined and continuous everywhere on C_1 as we have assumed that there are no characteristic points inside C_1.

Let U_2 be the (non-empty) interior of C_1. Then for each $q \in U_2$ there exists a maximal integral curve $\theta_q(t)$ of V, defined on some open interval $(\alpha(q), \beta(q))$ containing 0, such that $\theta_q(0) = q$.

Define the set

$$D = \left\{(0, y, z) \in \mathbb{H} \;\middle|\; -\frac{K}{L}a < z < \frac{K}{L}a,\ \alpha(\varphi(z)) < y < \beta(\varphi(z))\right\}$$

On this set the map $(0, y, z) \mapsto \theta_{\varphi(z)}(y)$ is well-defined. The set D is open and contains **0**, and so there exists $n \leq \frac{K}{2L}$ such that the set

$$C = 0 \times [-n, n] \times [-n, n]$$

is a subset of D.

We define our map $\Psi : C \to \mathbb{H}$ by the formula

$$\Psi(0, y, z) = \theta_{\varphi(z)}(y).$$

Note that by definition $f(\Psi(0, y, z)) = 0$. We next prove that this is a Lipschitz map from a compact subset of the level set $x = 0$ to the level set $f(x, y, z) = 0$.

Since there exists L such that $d_N((y_1, z_1), (y_2, z_2)) \geq L((y_1 - y_2)^4 + (z_1 - z_2)^2)^{\frac{1}{4}}$, it suffices to show that there exists a constant $A \geq 0$ such that for all $(0, y_1, z_1)$ and $(0, y_2, z_2)$ in C, we have that

$$d_{\mathbb{H}}(\Psi(0, y_1, z_1), \Psi(0, y_2, z_2)) \leq A\left((y_1 - y_2)^4 + (z_1 - z_2)^2\right)^{\frac{1}{4}}.$$

We first use the triangle inequality to break up the left hand side:

$$\begin{aligned} d_{\mathbb{H}}(\Psi(0, y_1, z_1), \Psi(0, y_2, z_2)) \leq & d_{\mathbb{H}}(\Psi(0, y_1, z_1), \Psi(0, y_2, z_1)) \\ & + d_{\mathbb{H}}(\Psi(0, y_2, z_1), \Psi(0, y_2, z_2)) \end{aligned}$$

We deal with each term on the right hand side separately, showing that

$$d_{\mathbb{H}}(\Psi(0, y_1, z_1), \Psi(0, y_2, z_1)) \leq |y_1 - y_2|\sqrt{1 + M^2} \tag{2}$$

and that there exists a constant A_2 so that

$$d_{\mathbb{H}}(\Psi(0, y_2, z_1), \Psi(0, y_2, z_2)) \leq A_2|z_1 - z_2|^{\frac{1}{2}} \tag{3}$$

With these estimates in place, we can show that Ψ is Lipschitz: Let A_1 be the maximum of A_2 and $\sqrt{1+M^2}$. Then, putting these estimates together, we get that

$$\begin{aligned} d_{\mathbb{H}}(\Psi(0,y_1,z_1),\Psi(0,y_2,z_2)) &\leq d_{\mathbb{H}}(\Psi(0,y_1,z_1),\Psi(0,y_2,z_1)) \\ &\quad + d_{\mathbb{H}}(\Psi(0,y_2,z_1),\Psi(0,y_2,z_2)) \\ &\leq |y_1-y_2|\sqrt{1+M^2} + A_2\,|z_1-z_2|^{\frac{1}{2}} \\ &\leq A_1\left(|y_1-y_2| + |z_1-z_2|^{\frac{1}{2}}\right) \\ &\leq A_1\cdot 2^{\frac{3}{4}}\left((y_1-y_2)^4+(z_1-z_2)^2\right)^{\frac{1}{4}} \\ &\leq A_1\cdot 2^{\frac{3}{4}}\cdot B\cdot d_{\mathbb{H}}\left((0,y_1,z_1),(0,y_2,z_2)\right) \quad \text{by (1)} \end{aligned}$$

Setting $A = A_1\cdot 2^{\frac{3}{4}}\cdot B$ proves that Ψ is a Lipschitz map of C onto a neighborhood of the origin in S. Since this construction works for any noncharacteristic point, we can cover any compact subset of S by a finite number of such neighborhoods union a portion of the characteristic locus. By an exhaustion argument, we see that S is the union of countably many such neighborhoods and the characteristic locus, which is $\mathscr{H}^3_{cc}$-measure zero. In other words, S is countably N-rectifiable, proving theorem 1.

We devote the rest of the paper to proving the estimates 2 and 3.

Claim 2:

$$d_{\mathbb{H}}(\Psi(0,y_1,z_1),\Psi(0,y_2,z_1)) \leq |y_1-y_2|\sqrt{1+M^2}$$

To show claim 2, we note that the sub-Riemannian distance between $\Psi(0,y_1,z_1)$ and $\Psi(0,y_2,z_1)$ is bounded above by the length of any horizontal curve connecting these two points. One such curve is $\theta_{\varphi(z_1)}(y)$ for $y_1 \leq y \leq y_2$. Write this curve in component form by

$$\varphi_{z_1}(y) = (\gamma_1(y), y, \gamma_3(y)).$$

The length of this curve is

$$\text{Length} = \left|\int_{y_1}^{y_2}\sqrt{1+\gamma_1'(y)}\,\mathrm{d}y\right|,$$

which, since $\theta_{\varphi(z_1)}(y)$ is the flow of V, is bounded above by $|y_1-y_2|\sqrt{1+M^2}$. Thus we have that

$$d_{\mathbb{H}}(\Psi(0,y_1,z_1),\Psi(0,y_2,z_1)) \leq |y_1-y_2|\sqrt{1+M^2}$$

Claim 3: *There exists a constant A_2 so that*

$$d_{\mathbb{H}}(\Psi(0,y_2,z_1),\Psi(0,y_2,z_2)) \leq A_2|z_1-z_2|^{\frac{1}{2}}$$

First, we write the curve $\theta_{\varphi(z_1)}(y)$ for $0 \leq y \leq y_2$ in the component form $(\gamma_1(y), y, \gamma_3(y))$ and the curve $\theta_{\varphi(z_2)}(y)$ under the same bounds in the component form $(\eta_1(y), y, \eta_3(y))$. In terms of these components, we are trying to find a bound on the quantity

$$\left((\gamma_1(y_2)-\eta_1(y_2))^4 + \left(\gamma_3(y_2)-\eta_3(y_2)+\frac{y_2}{2}(\gamma_1(y_2)-\eta_1(y_2))\right)^2\right)^{\frac{1}{4}}$$

Clearly, it is sufficient to bound $|\gamma_1(y_2)-\eta_1(y_2)|$ and $|\gamma_3(y_2)-\eta_3(y_2)|$. To this end, we show an intermediate inequality.

Lemma 2. *For all* $y \in [-n, n]$

$$|\eta_1(y) - \gamma_1(y)| \leq \frac{L}{K}\left|\eta_3(y) - \gamma_3(y) + \frac{1}{2}y(\eta_1(y) - \gamma_1(y))\right|$$

Proof: First, we multiply the left hand side of this inequality by $\frac{K}{K}$:

$$|\eta_1(y) - \gamma_1(y)| = \frac{K}{K}\,|\eta_1(y) - \gamma_1(y)|$$

Next, let $\psi_X(t)$ be the integral curve of the vector field X starting at $(\gamma_1(y), y, \gamma_3(y))$. Since $Xf \geq K$ on the set on which we are working, we have that

$$|f(\psi_X(\eta_1(y) - \gamma_1(y)) - f(\psi_X(0)))| \geq K\,|\eta_1(y) - \gamma_1(y)|$$

Applying this inequality and using the fact that $f(\psi_X(0)) = 0$, we have that

$$\frac{K}{K}\,|\eta_1(y) - \gamma_1(y)| \leq \frac{1}{K}\,|f(\psi_X(\eta_1(y) - \gamma_1(y)))|$$

The vector field $X = \left(1, 0, \frac{y}{2}\right)$ is constant along any of its integral curves, and as such we get that

$$\begin{aligned}\psi_X(\eta_1(y) - \gamma_1(y))) &= \left(\gamma_1(y) + (\eta_1(y) - \gamma_1(y)), y, \gamma_3(y) - \frac{y}{2}\,(\eta_1(y) - \gamma_1(y))\right)\\ &= \left(\eta_1(y), y, \gamma_3(y) - \frac{y}{2}\,(\eta_1(y) - \gamma_1(y))\right)\end{aligned}$$

Thus we now have that

$$|\eta_1(y) - \gamma_1(y)| \leq \frac{1}{K}\left|f\left(\eta_1(y), y, \gamma_3(y) - \frac{y}{2}\,(\eta_1(y) - \gamma_1(y))\right)\right|$$

To finish this proof, we recall that on the set on which we are working, $\partial_z f \leq L$. This, along with the fact that $f\,(\eta_1(y), y, \eta_3(y)) = 0$, implies that

$$\begin{aligned}&\left|f\,(\eta_1(y), y, \eta_3(y)) - f\left(\eta_1(y), y, \gamma_3(y) - \frac{y}{2}\,(\eta_1(y) - \gamma_1(t))\right)\right|\\ &\qquad = \left|f\left(\eta_1(y), y, \gamma_3(y) - \frac{y}{2}\,(\eta_1(y) - \gamma_1(y))\right)\right| = G\end{aligned}$$

and so,

$$\begin{aligned}G &\leq L\left|\eta_3(y) - \left(\gamma_3(y) - \frac{y}{2}\,(\eta_1(y) - \gamma_1(y))\right)\right|\\ &\leq L\left|\eta_3(y) - \gamma_3(y) + \frac{y}{2}\,(\eta_1(y) - \gamma_1(y))\right|\\ &\leq L\left|\eta_3(y) - \gamma_3(y) + \frac{y}{2}\,(\eta_1(y) - \gamma_1(y))\right|\end{aligned}$$

We now substitute the right hand side of this inequality to get our result:

$$|\eta_1(y) - \gamma_1(y)| \leq \frac{L}{K}\left|\eta_3(y) - \gamma_3(y) + \frac{y}{2}(\eta_1(y) - \gamma_1(y))\right|$$

□

Second, we rewrite the quantity $\eta_3(y) - \gamma_3(y)$ using the fact that both $\theta_{\varphi(z_1)}(y)$ and $\theta_{\varphi(z_2)}(y)$ are horizontal:

$$\begin{aligned}
\eta_3(y) - \gamma_3(y) &= \eta_3(0) + \int_0^y \eta_3'(t)\,\mathrm{d}t - \gamma_3(0) - \int_0^y \gamma_3'(t)\,\mathrm{d}t \\
&= z_2 - z_1 + \int_0^y \eta_3'(t)\,\mathrm{d}t - \int_0^y \gamma_3'(t)\,\mathrm{d}t \\
&= z_2 - z_1 + \int_0^y \left(\frac{1}{2}\eta_1(t) - \frac{t}{2}\eta_1'(t)\right)\,\mathrm{d}t - \int_0^y \left(\frac{1}{2}\gamma_1(t) - \frac{t}{2}\gamma_1'(t)\right)\,\mathrm{d}t \\
&= z_2 - z_1 + \int_0^y (\eta_1(t) - \gamma_1(t))\,\mathrm{d}t - \frac{y}{2}(\eta_1(y) - \gamma_1(y))\Big]_0^y \\
&= z_2 - z_1 + \int_0^y (\eta_1(t) - \gamma_1(t))\,\mathrm{d}t - \frac{y}{2}(\eta_1(y) - \gamma_1(y)).
\end{aligned}$$

Substituting this into the inequality above, we get that

$$|\eta_1(y) - \gamma_1(y)| \leq \frac{L}{K}\left|z_2 - z_1 + \int_0^y (\eta_1(t) - \gamma_1(t))\,\mathrm{d}t\right|.$$

Next we break up the right hand side using the triangle inequality:

$$|\eta_1(y) - \gamma_1(y)| \leq \frac{L}{K}|z_2 - z_1| + \frac{L}{K}\left|\int_0^y (\eta_1(t) - \gamma_1(t))\,\mathrm{d}t\right|.$$

Thus, by Gronwall's lemma,

$$|\eta_1(y_0) - \gamma_1(y_0)| \leq \frac{L}{K}\mathrm{e}^{\frac{1}{2}}|z_2 - z_1|$$

Note, we use that $y \leq \frac{K}{2L}$.

Using this, we can get a bound on $|\eta_3(y) - \gamma_3(y)|$ as well:

$$\begin{aligned}
|\eta_3(y) - \gamma_3(y)| &= \left|z_2 - z_1 + \int_0^y (\eta_1(t) - \gamma_1(t))\,\mathrm{d}t - \frac{y}{2}(\eta_1(y) - \gamma_1(y))\right| \\
&\leq |z_2 - z_1| + \left|\int_0^y (\eta_1(t) - \gamma_1(t))\,\mathrm{d}t\right| + \left|\frac{y}{2}(\eta_1(y) - \gamma_1(y))\right| \\
&\leq |z_2 - z_1| + \frac{\mathrm{e}^{\frac{1}{2}}}{2}|z_2 - z_1| + \frac{\mathrm{e}^{\frac{1}{2}}}{4}|z_2 - z_1| \quad \text{by (2)} \\
&\leq \left(1 + \frac{3\mathrm{e}^{\frac{1}{2}}}{4}\right)|z_2 - z_1|
\end{aligned}$$

Thus we have that

$$\begin{aligned}
&\left((\gamma_1(y_2) - \eta_1(y_2))^4 + (\gamma_3(y_2) - \eta_3(y_2))^2\right)^{\frac{1}{4}} \leq \\
&\qquad \left(\left(\frac{\mathrm{e}^{\frac{1}{2}}L}{K}|z_2 - z_1|\right)^4 + \left(\left(1 + \frac{3\mathrm{e}^{\frac{1}{2}}}{4}\right)|z_2 - z_1|\right)^2\right)^{\frac{1}{4}} \\
&\qquad \leq \left(\frac{\mathrm{e}^2 L^4}{K^4}|z_2 - z_1|^2 + \left(1 + \frac{3\mathrm{e}^{\frac{1}{2}}}{4}\right)^2\right)^{\frac{1}{4}} |z_1 - z_2|^{\frac{1}{2}}
\end{aligned}$$

The quantity

$$\left(\frac{\mathrm{e}^2 L^4}{K^4}\left|z_2 - z_1\right|^2 + \left(1 + \frac{3\mathrm{e}^{\frac{1}{2}}}{4}\right)^2\right)^{\frac{1}{4}}$$

is bounded on our domain: we will call this bound A_2. Hence we have that

$$\left((\gamma_1(y_2) - \eta_1(y_2))^4 + (\gamma_3(y_2) - \eta_3(y_2))^2\right)^{\frac{1}{4}} \leq A_2\left|z_1 - z_2\right|^{\frac{1}{2}}$$

This completes the proof of the claim.

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